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Recent Advances in Chebyshev Rational Approximation on Finite and Infinite Intervals

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INTRODUCTION

Chebyshev was the first mathematician to publish fundamental results in rational approximation. Chebyshev's results appeared much prior to the publication of the Weierstrass approximation theorem. Even though rational approximation is as old as polynomial approximation, not much is known about it. Developing methods to obtain error estimates for approximation by rational functions of a given degree is much more difficult than the corresponding polynomial case. Very recently some new powerful methods have been developed. For finite intervals, the credit should go to Gončar [13h, i] and Newman [17a, d, e]. For infinite intervals, most of the credit should go to Erdös and Newman. A forthcoming paper of Newman and Reddy [17l] will discuss these methods and their scope.

We divide this article into three sections. Section 1 deals mainly with finite intervals. In this connection we also refer the interested reader to the excellent article of Gončar [13j]. Section 2 is concerned with results on infinite intervals. In Section 3 we propose a series of open problems. Some of these are very difficult and present available methods are hopeless to attack them successfully.

DEFINITION. Let $f(x)$ be a real function, continuous on $[0, \infty)$ and with no zero there. Then set

$$\begin{aligned}
\lambda_{m,n} = \lambda_{m,n}(1/f) &= \inf_{r_{m,n}\in\pi_{m,n}} \left\| \frac{1}{f} - r_{m,n} \right\|_{L_\infty[0,\infty)}, \\
R_n(f, [a, b]) &= \inf_{r_{n,n}\in\pi_{n,n}} \| f - r_{n,n} \|_{L_\infty[a,b]}, \\
\epsilon_k &= \inf_{P\in\pi_k^*} \| x^n - P(x) \|_{L_\infty[0,1]}, \qquad (1)\\
\theta_k &= \inf_{P,Q\in\pi_k^*} \left\| x^n - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,1]}, \\
E_n(f, [a, b]) &= \inf_{P\in\pi_n} \| f - P \|_{L_\infty[a,b]},
\end{aligned}$$

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where $\pi_{m,n}$ denotes the class of all rational functions of the form $r_{m,n}(x) = P_m(x)/Q_n(x)$; P_m and Q_n represent real polynomials of degrees at most m and n, respectively. π_n denotes the class of real polynomials of degree at most n. $\pi_k{}^*$ denotes the class of all real polynomials of degree at most k having only nonnegative coefficients $(1 \leqslant k < n)$.

Let $r > 0$, $s > 1$. Then $\xi(r, s)$ denotes the unique open ellipse in the complex plane with foci at $x = 0$ and $x = r$ whose semimajor and semiminor axes a and b satisfy $b/a = (s^2 - 1)/(s^2 + 1)$.

If $F(z)$ is an entire function, we denote

$$\widetilde{M_F}(r, s) = \sup\{|F(z)|: z \in \xi(r, s)\}, \qquad M(r) = \max_{|z|=r} |F(z)|.$$

As usual, the order ρ of an entire function $F(z)$ is

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

If $0 < \rho < \infty$, then the type τ and the lower type ω $(0 \leqslant \omega \leqslant \tau \leqslant \infty)$ of $F(z)$ are given by

$$\lim_{r\to\infty} {\sup \atop \inf} \frac{\log M(r)}{r^\rho} = {\tau \atop \omega}.$$

$c_1, c_2, c_3, \ldots, c_{10}, \ldots, c_{20}$ are positive constants which may be different on different occasions.

1

According to a well-known result of Bernstein [4a] there is a polynomial $P^*(x)$ of degree at most n such that

$$\| \, |x| - P^*(x)\|_{L_\infty[-1,1]} \leqslant c_1/n. \tag{2}$$

Further, he has established that for every polynomial $P(x)$ of degree at most n,

$$\| \, |x| - P(x)\|_{L_\infty[-1,1]} \geqslant c_2/n. \tag{3}$$

In 1964, Newman [17a] established the following remarkable result. For every $n \geqslant 4$ there is a rational function $r(x)$ of degree at most n for which

$$\| \, |x| - r(x)\|_{L_\infty[-1,1]} < 3e^{-n^{1/2}}. \tag{4}$$

On the other hand, he has shown that for every rational function $r(x)$ of degree at most $n \geqslant 4$,

$$\| \, |x| - r(x)\|_{L_\infty[-1,1]} \geqslant \tfrac{1}{2}e^{-9n^{1/2}}. \tag{5}$$

Soon after the publication of these results several Hungarian and Soviet mathematicians published work extending and improving the latter. In these investigations Newman's method played a very significant role.

Gončar [13h] has derived Newman's result in a much sharper form from an earlier result of Zolotarev. It is interesting to note that Zolotarev's result was published in 1877, 8 years prior to the publication of the Weierstrass approximation theorem. Gončar's theorem reads as follows:

$$e^{-(\pi 2^{1/2}+\epsilon)n^{1/2}} < R_n(|x|) < e^{-((\pi/2)-\epsilon)n^{1/2}}, \qquad n \geqslant n_0(\epsilon). \tag{6}$$

Later on, by a more refined technique, Bulanov [7a], working under the supervision of Dolzenko, established:

$$\lim_{n\to\infty} (R_n(|x|))^{1/n^{1/2}} = e^{-\pi}.$$

More precisely, for $n = 0, 1, 2, 3,\ldots$ and any $M > 0$,

$$R_n(|x|) \geqslant e^{-\pi(n+1)^{1/2}}$$

and

$$R_n(|x|) \leqslant e^{-\pi(n)^{1/2}(1-\Delta(n))}, \tag{7}$$

where $\Delta(n) = O(n^{-M/(4M+1)})$.

Recently Vjačeslavov [28] sharpened (7) as follows:

$$R_n(|x|) \leqslant Ane^{-\pi(n)^{1/2}}, \tag{8}$$

where A is an absolute constant.

In [17d] it has been shown that $x^{1/2}$ can be approximated on [0, 1] by rational functions of degree n having real nonnegative coefficients with an error $\leqslant 3e^{-2n^{1/2}}$. Now it is natural to ask how close can $(1-x)^{1/2}$ be approximated on [0, 1] by rational functions of degree n having real non-negative coefficients. In this connectoin Reddy [17r] proved the following: For all $n \geqslant 2$,

$$\left\| (1-x)^{1/2} - \left(\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{x}{4}\right)^k\right)^{-1} \right\|_{L_\infty[0,1]} \leqslant 4\left(\frac{\log n}{n}\right)^{1/2}.$$

Let $p(x)$ and $q(x)$ be polynomials of degrees at most n having only real, nonnegative coefficients; then

$$\left\| (1-x)^{1/2} - \frac{p(x)}{q(x)} \right\|_{L_\infty[0,1]} \geqslant \frac{1}{4n^{1/2}}.$$

It is interesting to know how close $|x|$ can be approximated on $[-1, 1]$ by reciprocals of polynomials of degree $\leqslant n$. In this connection Lungu [15] has obtained the following:

For every polynomial $P(x)$ of degree at most n,

$$\left\| |x| - \frac{1}{P(x)} \right\|_{L_\infty[-1,1]} \geqslant c_5 n^{-1}.$$

Newman and Reddy [17h] have proved that there is a polynomial $P^*(x)$ of degree at most n for which

$$\left\| |x| - \frac{1}{P^*(x)} \right\|_{L_\infty[-1,1]} \leqslant \frac{\pi^2}{2n}.$$

It will be of interest to obtain similar results on approximatoin to $|x|$ on $[-1, 1]$ by rational functions of the form P_m/Q_n. It is also worth investigating whether one can deduce the theorems of Bernstein, Newman, Lungu, and Newman and Reddy from a single result. If this can be done, it will be a spectacular achievement.

The following generalization of Newman's result is due to Szusz and Turan [26c]: If f is piecewise analytic on $[a, b]$, then $R_{n,n}(f) < e^{-c(n)^{1/2}}$, $c = c_f > 0$, $n \geqslant 1$. Gončar [13h] proved the following converse result: If f is piecewise infinitely differentiable (or, in particular, piecewise analytic) on Δ and if $R_n(f) < e^{-A_n(n)^{1/2}}$ for some sequence $n = n_i \uparrow \infty$ as $i \uparrow \infty$, where $A_n \to \infty$ as $n \to \infty$, then f is infinitely differentiable (or analytic) on Δ.

The following results are also due to Szusz and Turan [26a,b]:

(i) For any function which is convex and belongs to Lip 1 on $[-1, 1]$, and for $n = 1, 2, \ldots$, one can find a suitable rational function of the form P_n/Q_n for which

$$\left\| f - \frac{P_n}{Q_n} \right\|_{L_\infty[-1,1]} < A \frac{\log^4 n}{n^2},$$

where A is a positive constant.

(ii) If $f(t)$ is a k-times differentiable function for which $f^{(k)}(t)$ $(-1 < t < 1)$ is convex and satisfies a Lipschitz condition, then there exists a rational function $r(t)$ of degree $<n$ for which on $(-1, 1)$

$$|f(t) - r(t)| \leqslant c_2 n^{-k-2} (\log n)^{2(k+2)}.$$

Freud [12a|c,d] has improved some of the above results.

The following result is due to Bulanov [7b].

The smallest deviation $R_N(f)$ of a continuous convex function $f(x)$ $(x \in [a, b], -\infty < a < b < \infty)$ from rational functions of degree at most N satisfies the inequality

$$R_N(f) \leqslant CMN^{-1}(\log N)^2, \qquad N = 2, 3, 4, \ldots,$$

where C is an absolute constant and M is the maximum modulus of $f(x)$ on $[a, b]$.

Recently Abdugapparov [1b] has established the following:

If $f^{(p)}(x)$ ($p \geqslant 1$) is continuous and convex on [0, 1], then we have there

$$R_n(f) \leqslant C_p M_p n^{-p-2} \log^3 n, \qquad n = 2, 3, 4, \ldots,$$

where C_p depends only on p and $M_p = \mathrm{Max}_{0 \leqslant x \leqslant 1} |f^{(p)}(x)|$.

A similar result was obtained earlier by Freud [12a], under an additional condition concerning the modulus of continuity of $f^{(p)}(x)$.

Freud and Szabados have obtained several interesting results. We mention here one of them from [12g].

Let $f(x)$ be a real continuous function on [0, 1] and let

$$|f(x)| \leqslant M \qquad (0 \leqslant x \leqslant 1).$$

Further, let

$$0 = \xi_n^{(0)} < \xi_n^{(1)} < \cdots < \xi_n^{(1(n))} = 1, \quad \delta_n = \min_{1 \leqslant k \leqslant s(n)} (\xi^{(k)} - \xi^{(k-1)})$$
$$(n = 2, 4, 6, \ldots).$$

If there are polynomials $p_n^{(k)}(x)$ of degree $\leqslant n$ for which

$$\xi_n = \max_{1 \leqslant k \leqslant s(n)} \max_{\xi_n^{(k-1)} \leqslant x \leqslant \xi_n^{(k)}} |f(x) - p_n^{(k)}(x)| \neq 0 \qquad (n = 2, 4, 6, \ldots)$$

tends to 0 as $n \to \infty$, then for all $N \geqslant$ some N_0 there exists a rational function $R_N(x)$ of degree $\leqslant N$ for which

$$\max_{0 \leqslant x \leqslant 1} |f(x) - R_N(x)| \leqslant 7\epsilon_n,$$

where n is the greatest even number satisfying the inequality

$$s(n)(3n + m - 1) \leqslant N, \qquad \text{with} \qquad m = \left[\log^2 \frac{57M^2 n^2 s(n)}{\delta_n {\epsilon_n}^3}\right] + 1.$$

The following result is due to Popov [18a]. Let f be a real function whose rth derivative is of bounded variation in [0, 1]. Then for every positive integer k,

$$R_n(f, [0, 1]) \leqslant C_{k,r} \frac{V_0^1(f^{(r)}) \overbrace{\log \log \cdots \log n}^{k \text{ times}}}{n^{r+1}},$$

where $C_{k,r}$ is a constant depending only on k and r, and $V_0^1(f^{(r)})$ denotes the total variation of $f^{(r)}$ on [0, 1]. This result improves an earlier one by Freud [12a]. Quite recently, Popov has succeeded in replacing (log log $\cdots$ log n) by a constant.

The following result is due to Freud [12a]. If $f \in \text{Lip}\,\alpha$ for some $\alpha > 0$ and f is of bounded variation, then $R_n(f) = O(n^{-1} \log^2 n)$.

It is interesting to know how the rate of decrease of the error $R_n(f)$ is related to the structural properties of the function f. In this connection Gončar has obtained several interesting results.

The following are from Gončar [13a]:

(i) If

$$R_n(f, [a, b]) < cn^{-1-\delta}, \qquad \delta > 0,$$

then f is differentiable almost everywhere on $[a, b]$.

(ii) If

$$R_n(f, [a, b]) < cn^{-k-\delta}, \qquad \delta > 0, \tag{9}$$

where $k > 0$ is an integer, then f possesses an asymptotic kth derivative almost everywhere on $[a, b]$.

In [9b] Dolženko observed that (9) guarantees the existence, almost everywhere on $[a, b]$, of not only an asymptotic kth derivative, but also a kth derivative in the sense of Peano (local differential of the kth order) of the function f.

The following interesting result is also due to Dolzenko [9a]:

(i) If

$$\sum_{n=1}^{\infty} R_n(f, [a, b]) < \infty,$$

then f is absolutely continuous on the interval $[a, b]$.

(ii) For any sequence of constants $a_n > 0$ with $\sum a_n = \infty$, there exists a real function f with $R_n(f, [a, b]) < a_n$ which is absolutely continuous on $[a, b]$ but does not possess almost everywhere there an asymptotic derivative.

It is obvious that $R_n(f, [a, b]) \leqslant E_n(f, [a, b])$. It is also known that there exist functions f for which $R_n(f, [a, b])$ decreases to zero very fast while $E_n(f, [a, b])$ tends to zero much slower. One obvious example is $f(x) = |x|$, $a = -1$, $b = 1$. Now it is natural to ask whether there exist functions f for which

$$R_n(f, [a, b]) = E_n(f, [a, b]).$$

In this connection Dolženko [9f] has established the following: There exists a function f such that

$$R_{n_k}(f, [a, b]) = E_{n_k}(f, [a, b])$$

for an infinite number of indices n_k.

Finally, we mention one of the significant results of Gončar [13i]. Let f be continuous on [0, 1] and let $\omega(\delta)$ be the modulus of continuity of f there. Let us assume that in the circle $D = \{x: |x - 1| < 1\}$ there exists a bounded analytic function which coincides with f on [0, 1). Then

$$R_n(f, [0, 1]) = O(\rho_n), \qquad \rho_n = \inf_{1<t<\infty} [te^{-cn/t} + \omega(e^{-t})].$$

By using (4) and (5), Freud [12b] has shown that for each $n \geqslant 4$ there is a rational function $r^*(x)$ of degree at most n for which

$$\left\| \frac{|x|}{1+x^2} - r^*(x) \right\|_{L_\infty(-\infty,\infty)} \leqslant e^{-c_6(n)^{1/2}}, \tag{10}$$

while for every rational function $r(x)$ of degree at most n,

$$\left\| \frac{|x|}{1+x^2} - r(x) \right\|_{L_\infty(-\infty,\infty)} \geqslant e^{-c_7(n)^{1/2}}. \tag{11}$$

Quite recently Erdös and Reddy [10c] have shown that every positive continuous function f on $[0, \infty)$, with $\lim_{x\to\infty} f(x) = 0$, can be approximated uniformly by reciprocals of Müntz polynomials on $[0, \infty)$. Hence, it is natural to ask how close $|x|/(1+x^2)$ can be approximated on $(-\infty, \infty)$ by reciprocals of polynomials. In this connection Newman and Reddy [17h] have proved the following: There is a polynomial $P^*(x)$ of degree at most n for which

$$\left\| \frac{|x|}{1+x^2} - \frac{1}{P^*(x)} \right\|_{L_\infty(-\infty,\infty)} \leqslant \frac{c_1}{n^{1/2}}, \tag{12}$$

and for every polynomial $P(x)$ of degree at most $n - 2$;

$$\left\| \frac{|x|}{1+x^2} - \frac{1}{P(x)} \right\|_{L_\infty(-\infty,\infty)} \geqslant \frac{c_1}{n^{1/2}}. \tag{13}$$

By comparing (10) and (13) one sees that rational functions of degree n give much less error than the reciprocals of polynomials of degree $\leqslant n$ in approximating $|x|/(1 + x^2)$ on $(-\infty, \infty)$. We will see shortly that this is not the case in general.

Chebyshev showed in 1858 that x^{n+1} can be uniformly approximated on $[-1, 1]$ by polynomials of degree at most n with an error of exactly 2^{-n}. In fact Chebyshev polynomials originated with this beautiful result. Later on Chebyshev's student Zolotarev [16a, p. 41] extended the above result of Chebyshev as follows. The best uniform approximation of $x^{n+1} - \sigma x^n$ $(0 \leqslant \sigma \leqslant (n + 1) \tan^2[\pi/(2n + 2)])$ on $[-1, 1]$ by polynomials of degree at most $(n - 1)$ gives an error of $2^{-n}(1 + (\sigma/(n + 1)))^{n+1}$. For $\sigma = 0$ we get above result of Chebyshev. Erdös and Szegö [10k, p. 467] proved Zolotarev's result by a different method.

Recently we have shown [17m] that x^{n+1} can be approximated on [0, 1] by reciprocals of polynomials of degree n with an error $(\frac{2}{3})^n$ but not better than $(12)^{-n}$.

Recently I have conjectured that x^{n+1} can be approximated by rational functions of degree at most n with an error $\leqslant 4^{-n}$. Newman [17e] has proved by a very simple and ingenious method the following. Let S and n be positive integers. Then: (i) There is a real polynomial $P(x)$ of degree $\leqslant n$ and a real polynomial, $q(x)$, of degree $2S$ such that, throughout $[-1, 1]$,

$$\left| x^{n+1} - \frac{P(x)}{q(x)} \right| \leqslant 2^{-n} \binom{S+n-2}{S}^{-1}. \tag{14}$$

(ii) If $P(x)$ is a real polynomial of degree $\leqslant n$ and $q(x)$ a real polynomial of degree $\leqslant 2S$, then somewhere in $[-1, 1]$,

$$\left| x^{n+1} - \frac{P(x)}{q(x)} \right| \geqslant 2^{-3-n} \binom{S+n+2}{S}^{-1}. \tag{15}$$

By adopting a different approach the above estimates (14) and (15) of Newman have been strengthened by Reddy.

Recently I have noted that the above results of Newman have been obtained for certain special cases by Achieser [2, p. 279]. In fact, by adopting a technique of Chebyshev, Achieser proved the following:

Let $a_0 \neq 0, a_1, a_2, \ldots, a_n$ be given real numbers. Then for every $N > n$,

$$\min_{p_i, q_i} \max_{-1 \leqslant x \leqslant 1} \left| a_0 x^N + \frac{a_1 x^{N-1}}{2} + \cdots + \frac{a_n x^{N-n}}{2^n} - \frac{q_0 x^{N-1} - q_1 x^{N-2} + \cdots + q_{n-1}}{p_0 x^n + \cdots + p_n} \right| = \frac{|\lambda|}{2^{N-1}},$$

where λ is a zero of minimal absolute value of the polynomial

$$\begin{vmatrix} c_n - \lambda & c_{n-1} & \cdots & c_1 & c_0 \\ c_{n-1} & c_{n-2} - \lambda & \cdots & c_0 & 0 \\ c_{n-2} & \vdots & \cdots & \vdots & 0 \\ \vdots & & & & \vdots \\ c_1 & c_0 & \cdots & -\lambda & 0 \\ c_0 & 0 & \cdots & 0 & -\lambda \end{vmatrix}$$

with $c_m = \sum_{i=0}^{[m/2]} a_{m-2i} \binom{N-m+2i}{i}$, $m = 0, 1, 2, \ldots, n$.

Reddy [19j] has extended the above results of Chebyshev, Zolotarev, and Newman as follows:

Let σ be a real number, $|\sigma| \neq 1$. Then for every real polynomials $P(x)$ and $Q(x)$ of degrees at most $(n-1)$ and $2m$, respectively, we have

$$\left\| x^{n+1} - \sigma x^n - \frac{P(x)}{Q(x)} \right\|_{L_\infty[-1,1]} \geqslant \frac{|1 - |\sigma||}{(2m+2)\, e^{4m+n+2}},$$

$$\left\| x^{n+1} + x^n - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,1]} \geqslant \frac{1}{(2m+2)(2e)^{2m+n+1}}.$$

Given a real constant σ and positive integers m, n, there exist real polynomials $P(x)$ of degree $\leqslant n$ and $Q(x)$ of degree $2m$ such that

$$\left\| x^{n+2} - \sigma x^{n+1} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[-1,1]} \leqslant \frac{(1 + |\sigma|)\, 2^{-n}}{\binom{m+n-2}{m}}.$$

Let $0 < a < b$, $n \geqslant 1$. Then x^{2n} can be approximated uniformly in the intervals $[-b, -a]$, $[a, b]$ by polynomials of degree at most $2n - 2$ with an error $2^{-(2n-1)}(b^2 - a^2)^n$. Also, for every polynomail $P_{2n}(x)$ we have for these intervals

$$\| x^{2n+1} - P_{2n}(x)\| \geqslant a(b^2 - a^2)^n\, 2^{-2n+1}.$$

Similarly there is a polynomial $P^*(x)$ of degree at most $2n$ for which, on these intervals,

$$\| x^{2n+1} - P^*(x)\| \leqslant b(b^2 - a^2)^n\, 2^{-2n+1}.$$

Further, there is a rational function of the form $P_{2n-2}(x)/Q_{4s}(x)$ for which, on the intervals, $(s = 1,\ n \geqslant 4)$

$$\left\| x^{2n} - \frac{p_{2n-2}(x)}{Q_{4s}(x)} \right\| \leqslant 2^{-2n+1} f(a, b, n)(b^2 - a^2)^{n+2}\, (n + 2)^{-1}$$

where

$$f(a, b, n) = \left(2(n + 1)(b^2 + a^2)^2 - (b^2 - a^2) - 8(b^2 + a^2)b^2 + \frac{16a^4}{n+2}\right)^{-1}.$$

While for every rational function of that form, we have there

$$\left\| x^{2n} - \frac{p_{2n-2}(x)}{Q_{4s}(x)} \right\| \geqslant \frac{(b^2 - a^2)^n\, 2^{-2n+1}}{(2sn^{-1} + 1)} \binom{2s + 2n - 1}{2s}^{-1} \left[T_{2s}\left(\frac{b^2 + a^2}{b^2 - a^2}\right)\right]^{-1}.$$

Let $p_{2n-2}(x)$ and $q_{2n-2}(x) = \sum_{k=0}^{2n-2} b_k x^k$, $b_k \geqslant 0$ $(k \geqslant 0)$, be any even polynomials of degrees at most $2n - 2$. Then, on $[-1, 1]$,

$$\left\| x^{2n} - \frac{p_{2n-2}(x)}{q_{2n-2}(x)} \right\| \geqslant 2^{-2n+1}.$$

Remark. According to Chebyshev's result x^{2n} can be approximated uniformly by polynomials of degree $2n - 2$ on $[-1, 1]$ with an error 2^{-2n+1}. Hence the best approximating rational function of the form $P_{2n-2}(x)/Q_{2n-2}(x)$ with $Q(x)$ having only nonnegative coefficients is identical to the best approximating polynomial of degree $2n - 2$.

Let $p(x)$ and $q(x)$ be real polynomials of degrees at most $m \geqslant 1$. Then if $m < 2n$, we have on $[-1, 1]$,

$$\left\| x^{2n} - \frac{p(x)}{q(x)} \right\| \geqslant 2^{-1} \cdot e^{-2\pi(2mn)^{1/2}}.$$

It will be interesting to know how close x^{n+1} can be approximated by polynomials of degree n having only nonnegative real coefficients. In this connection Newman and Reddy [17k] have proved the following:

If $P_k(x) = dx^k$, $1 \leq k < n$, $d > 0$, and satisfies the assumption

$$n(1-d) = (n-k)(k/n)^{k/(n-k)} d^{n/(n-k)},$$

then $P_k(x)$ is the best uniformly approximating polynomial of degree k to x^n on $[0, 1]$. In fact we have

$$n\epsilon_k = (n-k)(k/n)^{k/(n-k)}(1-\epsilon_k)^{n/(n-k)}, \qquad \epsilon_k = \theta_k = 1-d.$$

Reddy has shown for $0 \leq x \leq 1$, $0 \leq x^n(\sum_{k=0}^{m}(1-x)^k)^{-1} - x^{n+1} \leq 3^{-1}m^m n^n (m+n)^{-m-n}$. $(m \geq 2)$.

We consider now the approximation of e^x. It is well known that from the results of Bernstein we get the following:

There is a polynomial $P^*(x)$ of degree at most n for which, on $[-1, 1]$,

$$\| e^x - P^*(x) \| \leq \frac{e}{2^n(n+1)!}.$$

But for every polynomial $P(x)$ of degree $\leq n$ we have, somewhere in $[-1, 1]$,

$$| e^x - P(x) | \geq \frac{1}{e2^n(n+1)!}.$$

Recently Newman [17c] proved the following:

There is a rational function of the form $P_m(x)/Q_n(x)$ for which

$$\left\| e^x - \frac{P_m(x)}{Q_n(x)} \right\|_{L_\infty[-1,1]} \leq \frac{6 \cdot 2^{-m-n} m!\, n!}{(m+n)!\,(m+n+1)!}.$$

Newman and Reddy [17j] have obtained:

Let $P(x)$ and $Q(x)$ be real polynomials of degrees at most m and n, respectively. Then

$$\left\| e^x - \frac{P_m(x)}{Q_n(x)} \right\|_{L_\infty[-1,1]} \geq \frac{2^{-2n-2m-4} e^{-(n+1)/2(m+n+2)}}{(3+2(2)^{1/2})^n (m+n+2)^{m+n+2}}.$$

Now we mention an interesting result of Somorjai [24]: Let $f(x)$ be a real continuous function defined on $[0, 1]$. Let $\lambda_1, \lambda_2, \lambda_3, \ldots$, be an infinite sequence of real numbers with $\lambda_n \to \infty$. Then for $n = 1, 2, \ldots$, there are real sequences $\{a_k^{(n)}\}_{k=1}^n$ and $\{b_k^{(n)}\}_{k=1}^n$ for which

$$\lim_{n\to\infty} \left\| f(x) - \frac{\sum_1^n a_k^{(n)} x^{\lambda_k}}{\sum_1^n b_k^{(n)} x^{\lambda_k}} \right\|_{L_\infty[0,1]} \to 0.$$

Recently Bak [3a], and Bak and Newman [3a′] have improved the last result significantly.

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Now we consider approximation on the positive real axis. The following result is due to Meinardus *et al.* [16b]:

Let $f(x)$ be a positive continuous function on $[0, +\infty)$, and assume that there exists a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ and a number $q > 1$ such that

$$\limsup_{n\to\infty}\left\{\left\|\frac{1}{f(x)} - \frac{1}{P_n(x)}\right\|_{L_\infty[0,\infty)}\right\}^{1/n} \leqslant \frac{1}{q} < 1.$$

Then there exists an entire function $F(z)$ of finite order with $F(x) = f(x)$ for all $x \geqslant 0$. In addition, for every $s > 1$, there exist constants $K = K(s, q) > 0$, $\theta = \theta(s, q) > 1$, and $r_0 = r_0(s, q) > 0$ such that

$$\widetilde{M_F}(r, s) \leqslant (K\|f\|_{L_\infty[0,r]})^{\theta} \qquad \text{for all} \quad r \geqslant r_0 .$$

If, for each $s > 1$,

$$\tilde{\theta}(s) = \limsup_{r\to\infty}\left\{\frac{\log \widetilde{M_F}(r, s)}{\log \|f\|_{L_\infty[0,r]}}\right\}.$$

when f is unbounded and $\tilde{\theta}(s) = 1$ otherwise, then the order of F is

$$\leqslant \inf_{s>1}\left\{\frac{\log \tilde{\theta}(s)}{\log[\frac{1}{2} + \frac{1}{4}(s + 1/s)]}\right\},$$

and this upper bound is in general best possible.

The following result is due to Reddy and Shisha [19t].

Let $f(x)$ be a positive continuous function defined on $[0, +\infty)$. If there exists a sequence of polynomials $\{q_n(x)\}_{n=0}^{\infty}$ whose coefficients are $\geqslant 0$, with all $q_n(0) > 0$ such that

$$\left\|\frac{1}{f(x)} - \frac{1}{q_n(x)}\right\|_{L_\infty[0,\infty)} \to 0,$$

then $f(x)$ is the restriction to $[0, \infty)$ of an entire function $\sum_{k=0}^{\infty} a_k z^k$ with all $a_k \geqslant 0$.

Let $f(x)$ be any nonvanishing real continuous function defined on $[0, \infty)$. If there exists a sequence of real polynomials $q_n(x)$ of the form $\sum_{k=0}^{n} a_k^{(n)} x^{\mu_k}$, where $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, and $\sum_{k=1}^{\infty} 1/\mu_k < \infty$, with

$$\left\|\frac{1}{f(x)} - \frac{1}{q_n(x)}\right\|_{L_\infty[0,\infty)} \to 0,$$

then $f(x)$ is the restriction to $[0, \infty)$ of an entire function. This result is due to Erdös and Reddy [10i].

Schönhage [22] has shown that e^{-x} can be approximated on $[0, \infty)$ by reciprocals of polynomials of degree n roughly like 3^{-n}; this improves an earlier result of Cody *et al.* [8]. Newman and Reddy have shown that these polynomials must be zero free on $[0, -\epsilon n]$ for each $\epsilon > 0$ satisfying $(1 + 8(\epsilon)^{1/2}) < \log 3$. In [16c] Meinardus and Varga have proved: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geqslant 0$ $(k \geqslant 1)$, $a_0 > 0$, be an entire function of order ρ $(0 < \rho < \infty)$ satisfying the further assumption that

$$0 < \lim_{r\to\infty} \log M(r)/r^{\rho} < \infty.$$

Then

$$\frac{1}{2^{2+(1/\rho)}} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{2^{1/\rho}}. \tag{16}$$

In [19b] Reddy has succeeded in replacing lim sup by lim inf in (16). The following is due to Meinardas *et al.* [16b]:

Let $F(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with $a_0 > 0$ and $a_k \geqslant 0$ for all $k \geqslant 1$. If there exist $A > 0$, $s > 1$, $\theta > 0$, and $r_0 > 0$ such that

$$\tilde{M}(r, s) \leqslant A(\|f\|_{L_\infty[0,r]})^{\theta} \qquad \text{for all} \quad r \geqslant r_0, \tag{*}$$

then there exist a sequence $\{P_n(x)\}_{n=0}^{\infty}$ $q \geqslant s^{1/(1+\theta)} > 1$ such that

$$\limsup_{n\to\infty} \left\{\left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)}\right\}^{1/n} = \frac{1}{q} < 1. \tag{17}$$

Remark. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, is an entire function of finite positive-order, type and lower type, then obviously (*) is valid; hence (17) holds.

By adopting a very simple approach Reddy [19g] has proved the following: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function of order ρ $(0 < \rho < \infty)$, type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then for all large n

$$\left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} a_k x^k} \right\|_{L_\infty[0,\infty)} \leqslant \exp\left(\frac{-\omega n}{e\rho\tau + \rho\omega}\right).$$

On the other hand Reddy [19d] has proved that under the same conditions, for every polynomial $P(x)$ of degree at most n (sufficiently large),

$$\left\| \frac{1}{f(x)} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \geqslant \left[4\left(\frac{2\tau}{\omega}\right)^{1/\rho} - 1\right]^{-2}.$$

Newman [17b] considered the approximation of e^{-x} on $[0, \infty)$ by general rational functions and proved the following:

Let $P(x)$ and $Q(x)$ be real polynomials of degree at most $(n-1)$. Then there is a point x in $[0, \infty)$ for which

$$\left| e^{-x} - \frac{P(x)}{Q(x)} \right| \geqslant (128O)^{-n}.$$

If we restrict $P(x)$ and $Q(x)$ to have only nonnegative coefficients, then we have (cf. [10a])

$$\left\| e^{-x} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \geqslant \frac{1}{4ne^{n+1}}.$$

The following results are due to Reddy [19k].

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function of order ρ $(1 \leqslant \rho < \infty)$, type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then there is a positive constant c such that for every real polynomials $P(x)$, $Q(x)$ whose respective degrees m, n are sufficiently large,

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \geqslant \frac{ce^{-[(0.3)m+(0.7)n]\tau\omega^{-1}}}{8^n 6^m (7\tau)^{n/\rho}\, \omega^{-n/\rho}}.$$

Also there is $c_1 > 0$ such that, for every $\epsilon > 0$, $\rho(0 < \rho < 1)$ we have for every $P(x)$, $Q(x)$ as above,

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \geqslant \frac{ce^{-A^\rho\tau}8^{-n}6^{-m}}{((7\tau/\omega\rho^2) + (\tau/\omega))^{n/\rho}},$$

where $A^\rho = [0.3m + 0.7n][\omega[1+\epsilon)]^{-1}$.

The following result is due to Erdös and Reddy [10j].

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function, satisfying

$$1 < \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 < \infty$$

and

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = \begin{matrix}\tau_l\\ \omega_l\end{matrix} \qquad (0 < \omega_l \leqslant \tau_l < \infty).$$

Then for every rational function of the form $P_n(x)/Q_n(x)$, we have

$$\liminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{n^{-1-\Lambda^{-1}}} \geqslant G,$$

where

$$G = \exp\left\{ -\left(\frac{2}{\omega_l}\right)^{1/\Lambda} \left[\tau_l - 1 + \left(\frac{2\tau_l}{\omega_l}\right)^{1/(\Lambda+1)} \right] \right\}.$$

By using a lemma of Gončar–Newman–Zolotarev [13j, p. 450–451], Reddy [19h] proved the following:

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function. Let constants k $(0 < k < 1)$, c (>1), and ϵ (>0) be such that, with $\theta = 1 + \epsilon + \pi^2(\log c)^{-1}/(\log k^{-1})$, we have

$$M((1-K)r) > \{M((1-K)r)\}^{\theta}, \qquad (**)$$

for all large r. Then, for all large n, if $P(x)$ and $Q(x)$ are real polynomials of degree at most n,

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \geqslant C^{-2n\theta}.$$

Remark. If $f(z)$ is of finite, positive order, type and lower type, then clearly (**) holds.

Given an arbitrary entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $a_0 > 0$ and $a_k \geqslant 0$ $(k \geqslant 1)$, how close can $1/f(x)$ be approximated by reciprocals of polynomials of degree $\leqslant n$ on $[0, \infty)$? Erdös and Reddy [10d,f] have answered this as follows: For each $\epsilon > 0$ there exist infinitely many n such that for a proper P_n,

$$\left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant \exp\left(\frac{-n}{(\log n)^{1+\epsilon}}\right).$$

The following result is also due to Erdös and Reddy [10e]. Let $f(z) = 1 + \sum_{k=1}^{\infty} z^k/d_1 d_2 \cdots d_k$, $d_{k+1} > d_k > 0$, $k = 1, 2, 3,\ldots$, be an entire function of order $\rho < 1$ and regular growth. Then for any $\epsilon > 0$, we have for all large n,

$$\frac{d_1 d_2 d_3 \cdots d_n}{2^{4n} d_n^{2(\rho+\epsilon)} d_{n+1} d_{n+2} \cdots d_{2n}} \leqslant \lambda_{0,2n-1} \leqslant \frac{d_1 d_2 \cdots d_n}{d_{n+1} d_{n+2} \cdots d_{2n}} \left(\frac{d_{2n+1}}{d_{2n+1} - d_{2n}}\right).$$

The following result is due to Meinardus *et al.* [16b]: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function satisfying

$$1 < \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = A + 1 < \infty,$$

and

$$0 < \omega_l = \liminf_{r\to\infty} \frac{\log M(r)}{(\log r)^{A+1}} \leqslant \limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{A+1}} = \tau_l < \infty.$$

Then

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = 0.$$

Further, (Reddy [19a,e]):

$$\exp\left(\frac{-\Lambda}{(\Lambda+1)[(\Lambda+1)\,\tau_t]^{1/\Lambda}}\right) \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{n^{-[1+\Lambda^{-1}]}} < 1.$$

The following result is due to Reddy [19f]. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function of zero order satisfying

$$0 < \Lambda < \liminf_{n\to\infty} \frac{\log\log M(r)}{\log\log r} < \infty.$$

Then there exists a sequence $\{P_n(x)\}_{n=0}^{\infty}$ for which

$$\lim_{n\to\infty} \left\{\left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)}\right\}^{1/n} = 0.$$

Erdös and Reddy [10e] have established the following: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function for which $0 \leqslant \Lambda < \infty$. Then for each $\epsilon > 0$,

$$\liminf_{n\to\infty} (\lambda_{0,n})^{n^{-[1+(\Lambda+\epsilon)^{-1}]}} < 1.$$

Turning to the particular case $f(x) = e^x$, Freud *et al.* [12e] have obtained: For every real polynomial $P(x)$ of degree n,

$$\left\| e^{-|x|} - \frac{1}{P(x)} \right\|_{L_\infty(-\infty,\infty)} \geqslant \frac{c_{10}}{n}, \tag{18}$$

and there is a polynomial $P^*(x)$ of degree n for which

$$\left\| e^{-|x|} - \frac{1}{P^*(x)} \right\|_{L_\infty(-\infty,\infty)} \leqslant \frac{c_{11}\log n}{n}. \tag{19}$$

They have shown that there is a rational function $r^*(x)$ of degree n for which

$$\| e^{-|x|} - r^*(x) \|_{L_\infty(-\infty,\infty)} \leqslant e^{-c_{12}(n)^{1/2}}, \tag{20}$$

and for every rational function $r(x)$ of degree n,

$$\| e^{-|x|} - r(x) \|_{L_\infty(-\infty,\infty)} \geqslant e^{-c_{13}(n)^{1/2}}. \tag{21}$$

In [19o] Reddy has established: Let $0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n$ be even integers, satisfying $\alpha_k - \alpha_{k-1} \geqslant k$. Then for all $n \geqslant 2$,

$$\left\| e^{-|x|} - \frac{\sum_{j=1}^{n} a_j x^{\alpha_j}}{\sum_{j=1}^{n} e^{(j\log n)/n} a_j x^{\alpha_j}} \right\|_{L_\infty(-\infty,\infty)} \leqslant \frac{20\log n}{n},$$

where

$$a_k^{-1} = \left(\frac{\log n}{n}\right)^{\alpha_1}\left(\frac{2\log n}{n}\right)^{\alpha_2-\alpha_1}\cdots\left(\frac{k\log n}{n}\right)^{\alpha_k-\alpha_{k-1}}.$$

In [12e] one can find extensions of (18)–(21) to a whole class of entire functions. Erdös *et al.* [10b] by developing some new techniques have established: There is a real polynomial $P^*(x)$ of degree n for which

$$\left\| xe^{-x} - \frac{1}{P^*(x)} \right\|_{L_\infty[0,\infty)} \leqslant \frac{c_{14}(\log n)}{n^2},$$

while for every real polynomial $P(x)$ of degree n,

$$\left\| xe^{-x} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \geqslant c_{15}\frac{(\log n)}{n^2}.$$

On the other hand, it is easy to see that xe^{-x} can be approximated on $[0, \infty)$ by rational functions $r(x)$ of degree at most n with an error $\leqslant\delta^n$ $(0 < \delta < 1)$. In the same paper Erdös *et al.* have obtained: There is a real polynomial $P^*(x)$ of degree at most n which,

$$\left\| (1+x)\,e^{-x} - \frac{1}{P^*(x)} \right\|_{L_\infty[0,\infty)} \leqslant c_1 e^{-(2n)^{2/3}},$$

while for every $\epsilon > 0$, every $P_n(x)$ with n sufficiently large satisfies

$$\left\| (1+x)\,e^{-x} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \geqslant e^{-(2n)^{2/3}(1+\epsilon)}.$$

Again, $(1+x)\,e^{-x}$ can be approximated on $[0, \infty)$ by rational functions of degree $\leqslant n$ with an error $\leqslant\alpha^n$ $(0 < \alpha < 1)$. An extension of these results will be found in [10b].

Because of the above investigations, it is natural to ask, how close can $|x|\,e^{-|x|}$ be approximated by reciprocals of polynomials on $(-\infty, \infty)$. Newman and Reddy [17g] have proved: There is a real polynomial $P^*(x)$ of degree n for which

$$\left\| |x|\,e^{-|x|} - \frac{1}{P^*(x)} \right\|_{L_\infty(-\infty,\infty)} \leqslant \frac{c_{17}(\log n)^2}{n},$$

while for every such polynomial $P(x)$,

$$\left\| |x|\,e^{-|x|} - \frac{1}{P(x)} \right\|_{L_\infty(-\infty,\infty)} \geqslant \frac{c_{11}(\log n)}{n}.$$

There is a rational function $r^*(x)$ of degree n for which

$$\| \, | x | \, e^{-|x|} - r^*(x) \|_{L_\infty(-\infty,\infty)} \leqslant e^{-c_{11}(n)^{1/2}},$$

while for every such rational function $r(x)$,

$$\| \, | x | \, e^{-|x|} - r(x) \|_{L_\infty(-\infty,\infty)} \geqslant e^{-c_{20}(n)^{1/2}}.$$

It is interesting to note that from (14) and (15) one can derive easily that

$$\lim_{n\to\infty} \{\lambda_{n,n}[(1+x)^{-n}]\}^{1/n} = (27)^{-1}.$$

D. J. Newman has proved that $\lim_{n\to\infty}\{\lambda_{0,n}(1+x)^{-n}\}^{1/n} = 4/27$, a conjecture of A. R. Reddy (see also [10h]).

Now we consider approximations by rational functions having zeros and poles only on the negative real axis.

Let s be a set of real numbers and denote by R_s the set of all rational functions having all zeros and poles in s. Also let $C[a, b]$ denote the space of real continuous functions on $[a, b]$ equipped with the sup-norm on $[a, b]$. Newman [17f] has proved that if s is dense in $[a, b]$ and, for some $\epsilon > 0$, $s - [a - \epsilon, b + \epsilon]$ is infinite, then R_s is dense in $C[a, b]$.

The follwoing result is due to Newman and Reddy [17i]: There is a real polynomial $P(x)$ of degree at most n having zeros only on the negative real axis for which, for all $n \geqslant 2$,

$$\left\| e^{-x} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \leqslant \frac{1}{ne},$$

while for every real polynomial $P(x)$ of degree $n \geqslant 2$,

$$\left\| e^{-x} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \geqslant \frac{1}{17e^2 n}.$$

In [17i] it has been shown further that there is a rational function $r^*(x)$ of degree n having zeros and poles only on the negative real axis for which, for all large n,

$$\| e^{-x} - r^*(x) \|_{L_\infty[0,\infty)} \leqslant n^{-c_{21}\log n}.$$

On the other hand, for every rational function $r(x)$ of degree n having zeros and poles only on the negative real axis,

$$\| e^{-x} - r(x) \|_{L_\infty[0,\infty)} \geqslant e^{-6(n)^{1/2}}.$$

In [21] Saff *et al.* have obtained the following: There is a rational function $r(x)$ of degree at most n having all its poles on the negative real axis such that

$$\limsup_{n\to\infty} \{\| e^{-x} - r(x)\|_{L_\infty[0,\infty)}\}^{1/n} \leqslant \tfrac{1}{2}.$$

The following results are also from Newman and Reddy [17i]. (i) There is a real polynomial $P(x)$ of degree at most n ($\geqslant 2$) having zeros only in the left-half plane for which,

$$\left\| e^{-x} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \leqslant 4n^{-2}.$$

(It is easy to replace n^{-2} by $4n^{-4}$ by using $1 + x + (x^2/2!) + (x^3/3!) + (x^4/4!)$ instead of $1 + x + (x^2/2!)$ in the proof [17i, Theorem 3].) (ii) For every rational function $r(x)$ of degree at most $n \geqslant 2$, having zeros and poles only in the left-half plane,

$$\| e^{-x} - r(x)\|_{L_\infty[0,\infty)} \geqslant e^{-5n^{2/3}}.$$

The following results are from Erdös *et al.* [10a]: (i) Let $p(x)$ be a real polynomial of degree at most $n \geqslant 2$ having only real negative zeros. Then

$$\left\| e^{-x} - \frac{1}{p(x)} \right\| \geqslant \frac{1}{4ne^5}, \qquad x = 0, 1, 2,\dots .$$

(ii) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function. Then there is a real polynomial $p(x)$ of degree at most $n \geqslant 2$, for which

$$\left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\| \leqslant \frac{2}{f(n)}, \qquad x = 0, 1, 2,\dots .$$

(iii) Let $0 = a_0 < a_1 < a_2$ be given. Let $f(x)$ be a continuous, non-vanishing and monotonic increasing function on $[0, \infty)$. Then there exists a sequence $\{P_{2n}(x)\}_{n=0}^{\infty}$ for which

$$\left| \frac{1}{f(x)} - \frac{1}{P_{2n}(x)} \right| \leqslant \frac{2}{f(a_n)}, \qquad n = 0, 1, 2,\dots .$$

(iv) Let $p(x)$ and $q(x)$ be real polynomials of degrees at most n.

$$\left| e^{-x} - \frac{p(x)}{q(x)} \right| \geqslant \frac{(e-1)^n e^{-4n} 2^{-7n}}{n(3 + 2(2))^{n-1}}, \qquad x = 0, 1, 2,\dots .$$

The following are from Reddy ([19i], Theorems 1, 2): (a) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function of order $\rho = 2$, type τ, and lower type ω $(1/25 \leqslant \omega \leqslant \tau < \infty)$ or of order ρ $(2 < \rho < \infty)$, type τ, and lower type ω ($0 < \omega \leqslant \tau < \infty$). Then it is not possible to find for $n = 0, 1, 2,\ldots$ exponential polynomials $\sum_{k=0}^{\infty} b^{(n)} e^{kx}$ ($b^{(n)} \geqslant 0$) for which

$$\liminf_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k^{(n)} e^{kx}} \right\|_{L_\infty[0,\infty)}^{\rho\omega/(n^2\tau)} \leqslant e^{-1}.$$

(b) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ ($k \geqslant 1$), be an entire function of order ρ ($1 \leqslant \rho < \infty$), type τ, and lower type ω ($0 < \omega \leqslant \tau < \infty$). Let $\phi(z)$ be a transcendental entire function with nonnegative real coefficients satisfying

$$0 < \lim_{r\to\infty} \frac{\log M_\phi(r)}{(\log r)^2} = \theta < 1.$$

Then for every $g_n(x) = \sum_{k=0}^{n} b_k^{(n)} \{\phi(k)\}^k$, with all $b_k^{(n)} \geqslant 0$, we have

$$\liminf_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k^{(n)} \{\phi(x)\}^k} \right\|_{L_\infty[0,\infty)}^{\rho/[n(\log n)(\log\log n)]} \geqslant e^{-\tau/\omega}.$$[1]

3

Open Problems

Q.1. Do there exist rational functions $r_n(x)$ of degree at most n for which

$$\| e^{-x} - r_n(x) \|_{L_\infty[0,\infty)} \leqslant 4^{-n}?$$

Q.2. Using polynomials $p_n(x)$ of degree at most n, having only non-negative real coefficients, how small can one make

$$\left\| e^{-x} - \frac{1}{p_n(x)} \right\|_{L_\infty[0,\infty)}?$$

Q.3. Do there exist rational functions of degree at most n having zeros and poles only on the negative real axis and a positive constant c for which

$$\| e^{-x} - r_n(x) \|_{L_\infty[0,\infty)} \leqslant e^{-c(n)^{1/2}}?$$

Q.4. How close can e^{-x^2} be approximated uniformly on $[0, \infty)$ by rational functions of degree n having zeros and poles only on the negative real axis?

Q.5. How close can e^{-x^2} be approximated by rational functions of degree n on $\{0, 1, 2,\ldots\}$?

[1] *Note added in proof.* Recently I have extended several of the above results concerning $|x|$ and x^n to the case of several real variables. For example I have shown that $x_1^{2n_1} \times x_2^{2n_2} \cdots x_k^{2n_k}$, where each x_j lies in $[-b, -a] \cup [a, b]$, can be approximated there by a polynomial $P(x_1, x_2, \cdots x_k)$ of total degree not exceeding $2\sum_{i=1}^{k} n_i - 2 = N$, with maximal error equal to $2^{-N+2-k}(b^2 - a^2)^{(N+2)/2}$. Further, I have shown that the error obtained by rational functions in several variables is less than the error obtained by the corresponding polynomial.

Q.6. Is it possible to approximate x^{2n+1} on $[-1, 1]$ by the ratio of two monotonic polynomials of degree at most n with an error $\leqslant (2+\epsilon)^{-n}$ for an $\epsilon > 0$?

Q.7. Is it possible to approximate x^n on $[0, 1]$ by reciprocals of real polynomials of degree at most n with an error $\leqslant 4^{-n}$?

Q.8. Obtain error bounds for the approximation of x^n on $[-1, 1]$ by rational functions of the form P_k/Q_m, where $0 \leqslant k \leqslant n-2$, $m \geqslant 0$.

Q.9. Obtain error bounds for the approximation of $|x|$ on $[-1, 1]$ by rational functions of the form P_k/Q_m, $k \neq m$.

Q.10. Do there exist rational functions $r_n(x)$ of degree n and a constant $c > 0$ for which

$$\left\| \frac{x}{1 + x\log(1+x)} - r_n(x) \right\|_{L_\infty[0,\infty)} \leqslant \frac{c}{n \log n} ?$$

Q.11. Do there exist real polynomials $p(x)$ of degree at most n and a constant $c > 0$, for which

$$\left\| \frac{e^x - 1}{e^{2x}} - \frac{1}{p_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant \frac{c \log n}{n^2} ?$$

Q.12. Find error bounds for the approximation $(1 + |x|)/e^{|x|}$ on $(-\infty, \infty)$ by reciprocals of real polynomials of degree $\leqslant n$.

Q.13. Find error bounds for the approximation of $(1 + x^{n+1})^{-1}$ on $[0, \infty)$ by rational functions of degree n.

Q.14. Find error bounds for the approximation of e^{x^2} on $[-1, 1]$ by rational functions of degree n.

Q.15. How close can $e^{x^{1/2}}$ be approximated on $[0, 1]$ by rational functions of degree n having zeros and poles only on the negative real axis?

Q.16. How close can $e^{x^{1/2}}$ be approximated on $[0, 1]$ by rational functions of degree n having only real, nonnegative coefficients?

Q.17. How close can $e^{x^{1/2}}$ be approximated on $[0, 1]$ by reciprocals of monotonic polynomials of degree n?

Q.18. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all $k \geqslant 0$, be an entire function of perfectly regular growth. Is it true that $\lambda_{0,n}^{1/n} \to \delta$, $0 < \delta < 1$?

Q.19. Let $f(x)$ be a real continuous function on $[0, \infty)$, and suppose there exists a sequence $\{P_n(x)\}_{n=0}^{\infty}$ for which

$$0 < \limsup_{n\to\infty} \left\{ \left\| \frac{1}{f} - \frac{1}{p_n} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} < 1.$$

Is it true that for every sequence $\{r_n(x)\}$ of rational functions of degree n there is a positive constant for which

$$0 < \liminf_{n\to\infty} \left\{ \left\| \frac{1}{f} - r_n(x) \right\|_{L_\infty[0,\infty)} \right\}^{1/n} < 1?$$

Q.20. How close can x^{n-1} be approximated on $[0, 1]$ by rational functions of degree n having zeros and poles only on the negative real axis?

Q.21. Is it possible to approximate $e^{x^{1/2}}$ on $[0, 1]$ by rational functions of degree n having only nonnegative real coefficients with an error better than $O(1/n)$?

Q.22. Is it possible to approximate e^{-x^2} on $[0, 1]$ by polynomials of degree n having only real zeros better than $O(1/n)$?

Q.23. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ $(k \geqslant 0)$, be an entire function of perfectly regular growth. Is it always true that the best rational approximation to f on $[0, 1]$ is much better than the best approximation there by polynomials of the same degree?

Q.24. How close can e^x be approximated on $[0, 1]$ by monotonic polynomials of degree $\leqslant n$?

Concluding Remarks

I would like to explain here why I have stated such very special open problems. It is well known that the methods developed to get error bounds for the approximation of $|x|$ on $[-1, 1]$ by polynomials of degree n and by rational functions of degree n have been exploited significantly to solve more general problems. In the words of Bernstein [4b]: "As happens in all fields of mathematics, general theorems and methods arise and are developed always in an attempt to solve some specific problems." Thus, even though the problems offered here are very special, the methods to be developed to solve them may be applicable to more general problems. Even today we do not know how close $1 + \sum_{k=1}^{\infty} (x^k/k^k)$ can be approximated uniformly on $[-1, 1]$ by rational functions of degree n. There are many problems like this which are easy to explain, but seem very hard to solve: Their solution may involve completely new methods, of great scope and power.

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